Grade 7/8 Math Circles

October 2/3/4/5

Constructable Numbers Problem Set

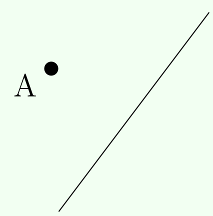
1. (a) Construct the numbers 3, 4, and 5.

   (b) Draw a triangle where two of the sides have length 3 and 5.

2. In this question, we will learn how to draw parallel lines.

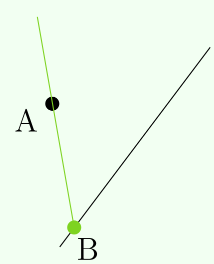
Example 1: Parallel Lines

Suppose we have the following line and point, labelled ‘A’.

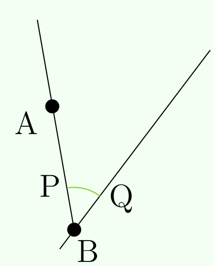


We’ll draw a parallel line that passes through the point.

(i) Pick a point on the line and label it ‘B’. Draw a new line through the points A and B.

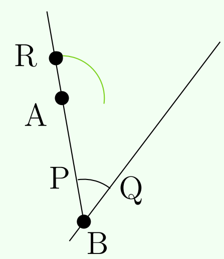


(ii) Using your compass, place the tip of the compass point B, and the other end on the line you just drew. Draw an arc from the new line to the old line.



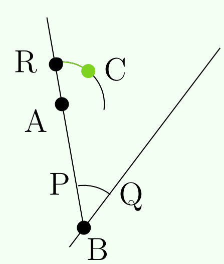
Label each end of the arc 'P' and 'Q'

(iii) Without adjusting your compass, place the tip on A and draw another arc.

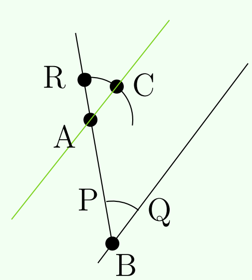


Label the point where the arc and line meet R.

(iv) Adjust your compass so one tip is on P and the other is on Q. Now, without adjusting the compass, place one tip on R. Place the other end so it is on the arc. Label this point on the second arc 'C'.



(v) Draw a line through A and C. This line is our parallel line!



(a) Practice drawing parallel lines by drawing a random line and point, and then drawing a parallel line.

(b) Re-do example 3, but instead of sliding your straightedge in step (v), draw a parallel line using the method that you just learned.

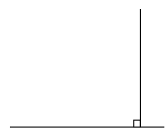
Caution: It's very easy for your paper to become crowded when drawing parallel lines at the same time as constructing a fraction. Try to erase any lines that you no longer need.

(c) Draw the fraction $\frac{5}{4}$ and use the above method to draw the parallel line in step (v) of constructing fractions.

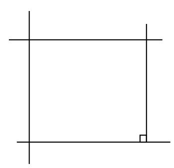
3. Construct the following fractions and use your constructions determine which pairs of fractions are equal.
(a) $\frac{5}{3}$ and $\frac{4}{2}$. (b) $\frac{1}{3}$ and $\frac{2}{6}$. (c) $\frac{1}{3}$ and $\frac{2}{3}$. (d) $\frac{3}{3}$ and 1.

4. Now that we can draw both perpendicular lines and parallel lines, draw a rectangle without using a protractor. Try following these steps:

(i) Following the steps from the lesson, draw a straight line and make a line perpendicular to it.

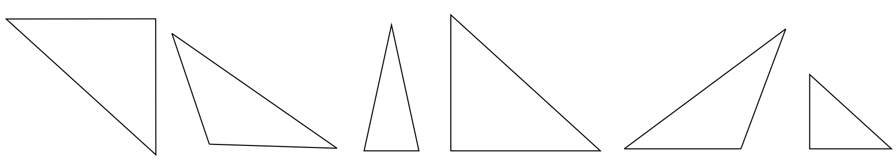


(ii) Following the steps in example 1, draw a line that's parallel to the horizontal line. Draw another line that's parallel to the vertical line.



Now you have a rectangle!

5. Find pairs of congruent triangles below:



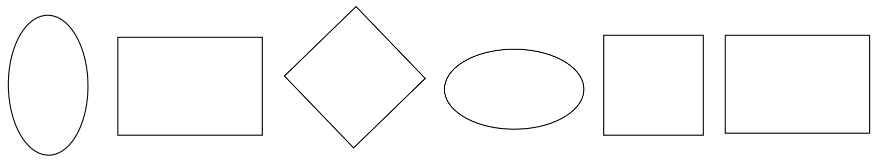
6. When talking about congruence, we don't have to only talk about triangles. Two shapes are congruent if you can do one or more of the following to go from one shape to the other.

(i) Shift the shapes,

(ii) Rotate the shapes,

(iii) Reflect the shapes.

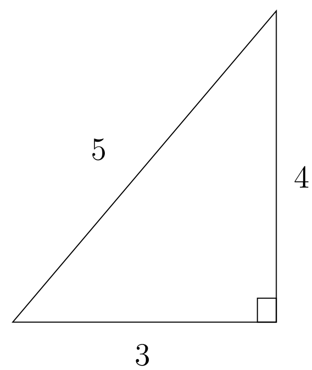
Which of the following shapes are congruent?



7. Kateryna is making a dance floor out of wood. She has a rectangular piece of wood that has side lengths 4 and 6 and she wants her floor to have a width of 3. Using geometry, how long will her dance floor be if she doesn't waste any wood? Make sure you accurately construct your rectangles using either the method from question 4 or by using a protractor.

Extension: Pythagorean Theorem

Consider the following right-angled triangle (not to scale)



Notice that the side lengths of the triangle have the following relation:

$$3^2 + 4^2 = 5^2$$

This is in fact true anytime that we have a right-angled triangle. If we replace 3 with a, 4 with b, and 5 with c, then all right-angled triangles satisfy

$$a^2 + b^2 = c^2$$

We call c the *hypotenuse* of the triangle and a and b the *legs* of the triangle.

Exercise 1

(i) Identify which numbers are the legs, and which number is the hypotenuse in the above triangle.

(ii) Suppose we have a right-angled triangle with legs of length 5 and 12. Using the *Pythagorean Theorem*, what is c^2? What is c?

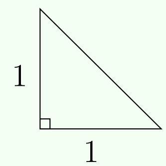
Hint: To find out what c is, take the *square root* of c^2. The symbol for square root on your calculator is $\sqrt{}$

(iii) Repeat part (ii) using legs with length 8 and 15.

Since we can construct right-angled triangles using just a straightedge and compass, whatever number we get for c, the length of the hypotenuse, is also constructable.

Example 2

Using your straightedge and compass, draw a right-angled triangle with legs both equal to 1.



Using the Pythagorean Theorem (and your calculator), calculate the length of the hypotenuse. You should get

$$c^2 = 2 \qquad\qquad c = \sqrt{2} = 1.414213562...$$

Notice that c doesn't look like a whole number or a fraction! And in fact, it isn't. We've discovered a new type of number, called an *irrational number*. Irrational numbers have never ending, never repeating decimals.

The special case of irrational numbers made using square roots are always constructable numbers, since we can draw a right triangle and use the Pythagorean Theorem.

Just like with whole numbers and fractions, once we've constructed one of these irrational numbers, we never have to construct it again.

Exercise 2

Try the following problems:

1. Draw a right-angled triangle with legs 1 and 2 to construct the number $\sqrt{5}$. Use the Pythagorean Theorem to verify that the we should have $\sqrt{5}$ as the hypotenuse.
2. What number will you construct if you draw a right triangle with legs of length 4 and 2? What about 3 and 1?
3. **(Challenge!)** Construct the fraction $\frac{\sqrt{2}}{2}$. Construct the fraction $\frac{1}{\sqrt{2}}$. Are these two numbers the same?